CANONICAL VARIATION OF A LORENTZIAN METRIC

BENJAMÍN OLEA

ABSTRACT. Given a Lorentzian manifold (M, g_L) and a timelike unitary vector field E, we can construct the Riemannian metric $g_R = g_L + 2\omega \otimes \omega$, being ω the metrically equivalent one form to E. We relate the curvature of both metrics, especially in the case of E being Killing or closed, and we use the relations obtained to give some results about (M, g_L).

1. INTRODUCTION

Given a Lorentzian manifold (M, g_L) and a timelike unitary vector field $E \in \mathfrak{X}(M)$ we can construct the Riemannian metric

$$(1) \qquad g_R = g_L + 2\omega \otimes \omega,$$

being ω the g_L-metrically equivalent one form to E. This construction is frequently used to exploit the positive definiteness of g_R, which provides some conclusions about g_L. For example, in [8] it is used to prove, under suitable conditions, the existence of periodic timelike geodesics in a compact Lorentzian manifold and in [3] to give a Bernstein theorem for lightlike hypersurfaces in $\mathbb{R}^n_1$. A similar construction has also been used to induce a Riemannian metric on a lightlike hypersurface of a Lorentzian manifold, which allows to define its extrinsic scalar curvature, [1]. Some aspects of Lorentzian metrics constructed in this way haven been studied in [21] and [22].

The dual construction of (1), i.e., given a Riemannian manifold (M, g_R) define the Lorentzian metric

$$(2) \qquad g_L = g_R - 2\omega \otimes \omega,$$

is also interesting because it provides some important examples of Lorentzian manifold, [11, 14, 25].

In this paper, we consider a Riemannian or a Lorentzian manifold (M, g) and a (timelike in the Lorentzian case) unitary vector field $E \in \mathfrak{X}(M)$. We call $\varepsilon = g(E, E)$ and define, for $t \neq -\varepsilon$,

$$g_t = g + t\omega \otimes \omega.$$

This metric is Lorentzian if $t < -\varepsilon$ and Riemannian if $-\varepsilon < t$. We call it *the canonical variation of g along E* due to its analogy with the canonical variation of a Riemannian submersion, where the metric of the fibres is multiplied by a parameter t, [5]. In [23], the metric g_t is also called variation of g and it is used to

---

contruct a Riemann metric with strictly positive sectional curvature from another Riemann metric with nonnegative sectional curvature.

Obviously, the construction (1) corresponds to $t = 2$ and $\varepsilon = -1$ and (2) to $t = -2$ and $\varepsilon = 1$ and we call them *standard canonical variation*.

Observe that metrics g_R and g_L in (1) and (2) are related in the same way and therefore it would be sufficient just to study, for example, the metric g_R constructed in (1) to obtain analog results for g_L constructed in (2). Nevertheless, the introduction of the parameters t and ε allows us to handle jointly (1) and (2), obtaining formulas easily adaptable to each case.

We use X, Y, Z letters for othogonal vector fields to E and U, V, W for arbitrary vector fields. We write the geometric objects derived from g_t with a t subscript, except for the connection which will be denoted by ∇^t. For example K_t, Ric_t and S_t are the sectional, Ricci and scalar curvature of g_t respectively. When we deal with g_L, g_R or g we use a L, R or no subscript respectively.

In the second section, we show a formula relating the difference tensor of the connections ∇^t and ∇, the exterior differential of ω and the Lie derivative $L_E g$. We also introduce the notion of vector field with *normal associated endomorphism*, which extends the notion of closed or conformal vector field. This concept will be useful to simplify the computation of the curvature of the canonical variation that are made in the third section. In section 4 we consider the standard canonical variation along a Killing unitary vector field and we obtain some inequalities about the curvature. We use the Berger theorem to show that if there exists a timelike unitary Killing vector field in a compact Lorentzian manifold with negative sectional curvature on timelike planes, then it has odd dimension. We also use Bochner techniques to give an integral inequality in a compact Lorentzian manifold furnished with a timelike Killing vector field. In the fifth section we give a result about the geodesic completeness of the canonical variation along a closed vector field and in the last section we consider how a lightlike hypersurface is transformed under standard canonical variation. We also give some sufficient conditions for a compact lightlike hypersurface to be totally geodesic.

2. Preliminaries

From now on, let (M, g) be a Riemannian or a Lorentzian manifold, $E \in \mathfrak{X}(M)$ a (timelike in the Lorentzian case) unitary vector field, ω its metrically equivalent one form and $\varepsilon = g(E, E)$.

Definition 2.1. *Fixed $t \in \mathbb{R} - \{-\varepsilon\}$, the canonical variation of g along E is defined as*

$$(3) \qquad\qquad\qquad g_t = g + t\omega \otimes \omega.$$

If $t = -2\varepsilon$, then it is called the standard canonical variation of g.

Example 2.2. *We give some examples of the standard canonical variation of a Riemannian manifold.*

   (1) *The standard canonical variation along any parallel vector field in the Euclidean space gives us the Minkowski space.*
   (2) *Consider the hyperbolic space $\mathbb{H}^n = \left(\mathbb{R}^n, dx_1^2 + e^{2x_1} \sum_{i=2}^n dx_i^2 \right)$. The standard canonical variation along $E = \partial_{x_1}$ gives us $-dx_1^2 + e^{2x_1} \sum_{i=2}^n dx_i^2$, which is a piece of $\mathbb{S}_1^n$. On the other hand, if we take $E = e^{-x_1} \partial_{x_n}$, the*

standard canonical variation is $dx_1^2 + e^{2x_1} \left(\sum_{i=2}^{n-1} dx_i^2 - dx_n^2 \right)$, which is a piece of $\mathbb{H}_1^n$.

(3) Fix $v \in \mathbb{R}^n$ and let $f : \mathbb{S}^{n-1} \to \mathbb{R}$ be given by $f(p) = p \cdot v$. Take $\mathbb{S}_+^{n-1} = \{p \in \mathbb{S}^{n-1} : f(p) > 0\}$ with its induced metric g_0. Since $Hess_f = -fg_0$, we have that $\left(\mathbb{S}_+^{n-1} \times \mathbb{R}, g_0 + f^2 dt^2 \right)$ is an open set of $\mathbb{S}^n$. The standard canonical variation along $E = \frac{1}{f} \partial_t$ is the piece of $\mathbb{S}_1^n$ given by $\left(\mathbb{S}_+^{n-1} \times \mathbb{R}, g_0 - f^2 dt^2 \right)$.

(4) The Lorentzian Berger spheres are obtained as the standard canonical variation along the Hopf vector field of the Euclidean spheres $\mathbb{S}^{2n+1}$.

(5) The Lie group $SL_2(\mathbb{R})$ can be furnished with the bi-invariant Lorentzian metric given by $< X, Y >_L = \frac{1}{2} tr(XY)$ or with the Riemannian left-invariant metric given by $< X, Y >_R = \frac{1}{2} tr(XY^t)$, being $X, Y \in \mathfrak{sl}_2(\mathbb{R})$. If we call

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}),$$

then it is easy to show that $<>_R$ is the standard canonical variation of $<>_L$ along E. Recall that $(SL_2(\mathbb{R}), <>_L)$ is isometric to $\mathbb{H}_1^3$.

We call $D^t = \nabla^t - \nabla$ the difference tensor. We can relate this tensor, $L_E g$ and ω as follows.

Proposition 2.3. *Given* $U, V, W \in \mathfrak{X}(M)$, *it holds*

(4) $g_t(D^t(U,V), W) = \frac{t}{2} \Big(\omega(W) (L_E g)(U,V) + \omega(U) d\omega(V,W) + \omega(V) d\omega(U,W) \Big).$

Proof. We can suppose that Lie brackets vanish. By the Koszul formula and equation (3) we have

$$
\begin{aligned}
2g_t(\nabla_U^t V, W) &= 2g(\nabla_U V, W) + t \big(U \big(\omega \otimes \omega(V,W) \big) + V \big(\omega \otimes \omega(U,W) \big) \\
&\quad - W \big(\omega \otimes \omega(U,V) \big) \\
&= 2g(\nabla_U V, W) + t \big(\nabla_U(\omega \otimes \omega)(V,W) + \nabla_V(\omega \otimes \omega)(U,W) \\
&\quad - \nabla_W(\omega \otimes \omega)(U,V) \big) + 2t\omega \otimes \omega(W, \nabla_U V).
\end{aligned}
$$

Using again equation (3),

$$g_t(D^t(U,V), W) = \frac{t}{2} \Big(\nabla_U(\omega \otimes \omega)(V,W) + \nabla_V(\omega \otimes \omega)(U,W) - \nabla_W(\omega \otimes \omega)(U,V) \Big).$$

Now, since $d\omega(U,V) = (\nabla_U \omega)(V) - (\nabla_V \omega)(U)$ and $(L_E g)(U,V) = (\nabla_U \omega)(V) + (\nabla_V \omega)(U)$, the right hand of the above expression is

$$\frac{t}{2} \Big(\omega(W) (L_E g)(U,V) + \omega(V) d\omega(U,W) + \omega(U) d\omega(V,W) \Big).$$

$\square$

We have the following consequences.

Corollary 2.4. *Take* $V, W, X, Y \in \mathfrak{X}(M)$ *with* $X, Y \in E^\perp$.

(1) $g(D^t(X,V), X) = 0.$

(2) $g(D^t(V,E), E) = 0.$

(3) $g_t(D^t(V,E), W) + g_t(D^t(W,E), V) = t \big(\omega(V) g(W, \nabla_E E) + \omega(W) g(V, \nabla_E E) \big).$

(4) $\nabla_E^t E = (1 + \varepsilon t) \nabla_E E.$

(5) $D^t(X,Y) = \frac{t}{2(1+\varepsilon t)} (L_E g)(X,Y) E.$

(6) $(L_E g_t)(X,Y) = (L_E g)(X,Y)$. *In particular, if E is orthogonally confor-
mal for g, then it is also orthogonally conformal for g_t.*
(7) $div_t V = div V$.

We need to introduce the following concepts for next sections.

Definition 2.5. *The associated endomorphism to a vector field $U \in \mathfrak{X}(M)$ is $A_U : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $A_U(V) = \nabla_V U$.*

Let A_U^* be the adjoint endomorphism of A_U. If $g(A_U(V), A_U(V)) = g(A_U^*(V), A_U^*(V))$ for all $V \in \mathfrak{X}(M)$, or equivalently, A_U and A_U^* commutes, then A_U is called normal.

Definition 2.6. *Let $E \in \mathfrak{X}(M)$ be a unitary vector field. A_E is orthogonally normal if $g(A_E(X), A_E(X)) = g(A_E^{*\perp}(X), A_E^{*\perp}(X))$ for all $X \in E^\perp$, being $A_E^{*\perp}(X)$ the orthogonal component to E.*

If A_E is orthogonally normal, then it is easy to show that A_E is normal if and only if E is geodesic. Moreover, if $U \in \mathfrak{X}(M)$ with A_U normal and E is its unitary, then A_E is orthogonally normal.

Example 2.7. *We give some examples of vector field with normal associated endomorphism.*

(1) *The associated endomorphism of a closed vector field is normal.*
(2) *If U is a conformal vector field, then $A_U^* = 2\rho \cdot id - A_U$ for certain $\rho \in C^\infty(M)$ and thus it is straightforward to check that A_U is normal.*
(3) *A_U is normal if and only if its associated matrix respect to a frame field is normal (it commutes with its transpose). Using this, it is easy to check that $U = (x+y)\partial_x + (y+z)\partial_y + (x+z)\partial_z$ has normal associated endomorphism in $\mathbb{R}^3$ and so its unitary E has orthogonally normal associated endomorphism in $\mathbb{R}^3 - \{0\}$. Observe that U is not closed neither conformal.*
(4) *Let G be a Lie group with a bi-invariant metric g. If U is any left-invariant vector field, then $A_U^*(X) = -A_U(X)$ and hence it is normal.*

Proposition 2.8. *Let (M,g) be a semi-Riemannian manifold, $U \in \mathfrak{X}(M)$ a vector field with normal associated endomorphism and $\overline{M}$ a nondegenerate hypersurface of M with unitary normal N. The projection of U onto $\overline{M}$ has normal associated endomorphism if and only if*

(5)
$$g(A_U(X), N)^2 - g(X, A_U(N))^2 = 2g(U,N)\big(g(A_U(X), S(X)) - g(A_U(S(X)), X)\big)$$

for all $X \in \mathfrak{X}(\overline{M})$, being S the shape operator of $\overline{M}$.

Proof. Take $\delta = g(N,N) = \pm 1$ and write $U = V + \delta g(U,N)N$ with $V \in \mathfrak{X}(\overline{M})$. Call $A_V(X) = tan(\nabla_X V)$, where $tan(\cdot)$ denotes the projection onto $\overline{M}$. Given $X \in \mathfrak{X}(\overline{M})$,

$$A_V(X) = tan(A_U(X)) - \delta g(U,N)\nabla_X N,$$
$$A_V^*(X) = tan(A_U^*(X)) - \delta g(U,N)\nabla_X N.$$

Therefore,

$$g(A_V(X), A_V(X)) = g(tan(A_U(X)), tan(A_U(X))) - 2\delta g(U, N)g(\nabla_X N, A_U(X))$$
$$+ g(U, N)^2 g(\nabla_X N, \nabla_X N)$$
$$= g(A_U(X), A_U(X)) - \delta g(A_U(X), N)^2 - 2\delta g(U, N)g(\nabla_X N, A_U(X))$$
$$+ g(U, N)^2 g(\nabla_X N, \nabla_X N).$$

Analogously,

$$g(A_V^*(X), A_V^*(X)) = g(A_U^*(X), A_U^*(X)) - \delta g(A_U^*(X), N)^2 - 2\delta g(U, N)g(\nabla_X N, A_U^*(X))$$
$$+ g(U, N)^2 g(\nabla_X N, \nabla_X N).$$

Hence, A_V is normal if and only if equation (5) holds. $\qquad\square$

Example 2.9. *Above proposition provides more examples of vector fields with normal associated endomorphism.*

(1) *If U is tangent to $\overline{M}$ and it has normal associated endomorphism, then the restriction of U to $\overline{M}$ also has normal associated endomorphism since equation (5) holds trivially.*

(2) *The projection of a Killing vector field onto an umbilic hypersurface has normal associated endomorphism, since equation (5) holds in this case. Observe that the projection of a Killing vector field onto a hypersurface is also a Killing vector field if and only if the hypersurface is totally geodesic.*

3. Curvature of the canonical variation

To relate the curvature tensor R of a metric g and the curvature tensor R^t of its canonical variation g_t we need the following general lemma.

Lemma 3.1. *Let ∇ and ∇^t be two arbitrary connections on a manifold M with curvature tensors R and R^t respectively. Given $U, V, W \in \mathfrak{X}(M)$ we have*

$$\begin{aligned} R_{UV}^t W &= R_{UV}W + (\nabla_U D^t)(V, W) - (\nabla_V D^t)(U, W) \\ &\quad + D^t(U, D^t(V, W)) - D^t(V, D^t(U, W)). \end{aligned}$$

(6)

Theorem 3.2. *Let (M, g) be a Riemannian or a Lorentzian manifold and g_t the canonical variation along a (timelike) unitary vector field E with A_E orthogonally normal. Given $X \in E^\perp$*

$$\begin{aligned} g_t(R_{XE}^t E, X) &= g(R_{XE}E, X) + t\left(\varepsilon g(\nabla_X A_E(E), X) - g(A_E(E), X)^2\right) \\ &\quad + \frac{t(2\varepsilon + t)}{2}\left(g(A_E(X), A_E(X)) - g(A_E^2(X), X)\right). \end{aligned}$$

Proof. Applying equation (6), we have

$$\begin{aligned} g(R_{XE}^t E - R_{XE}E, X) &= g((\nabla_X D^t)(E, E), X) - g((\nabla_E D^t)(X, E), X) \\ &\quad + g(D^t(X, D^t(E, E)), X) - g(D^t(E, D^t(X, E)), X). \end{aligned}$$

We compute each term applying proposition 2.3. The first one is $g((\nabla_X D^t)(E, E), X) = \varepsilon tg(\nabla_X \nabla_E E, X) - \varepsilon td\omega(\nabla_X E, X)$. For the second one we use corollary 2.4.

$$\begin{aligned} g((\nabla_E D^t)(X, E), X) &= -g_t(D^t(X, E), \nabla_E X) - g_t(D^t(\nabla_E X, E), X) \\ &= -tg(E, \nabla_E X)g(\nabla_E E, X) \\ &= tg(\nabla_E E, X)^2. \end{aligned}$$

6 BENJAMÍN OLEA

The third one is zero since $D^t(E,E) \perp E$. The last one.

$$
\begin{aligned}
g(D^t(E, D^t(X,E)), X) &= \frac{\varepsilon t}{2} d\omega(D^t(X,E), X) \\
&= \frac{\varepsilon t}{2} \left(g(\nabla_{D^t(X,E)} E, X) - g_t(D^t(X,E), \nabla_X E) \right) \\
&= \frac{\varepsilon t}{2} \left(g(\nabla_{D^t(X,E)} E, X) - \frac{\varepsilon t}{2} d\omega(X, \nabla_X E) \right).
\end{aligned}
$$

Now, we have $d\omega(A_E(X), X) = g(A_E^2(X), X) - g(A_E(X), A_E(X))$ and

$$
g(A_E(D^t(X,E)), X) = \frac{\varepsilon t}{2} \left(g(A_E(X), A_E^{*\perp}(X)) - g(A_E^{*\perp}(X), A_E^{*\perp}(X)) \right).
$$

Therefore, since A_E is orthogonally normal, $\left(\varepsilon + \frac{t}{4}\right) d\omega(\nabla_X E, X) + \frac{\varepsilon}{2} g(\nabla_{D^t(X,E)} E, X) = \frac{2\varepsilon + t}{2} \left(g(A_E^2(X)), X) - g(A_E(X), A_E(X)) \right)$ and we obtain the desired result. $\square$

Corollary 3.3. *Let (M,g) be a Riemannian or Lorentzian surface and g_t the canonical variation of g along a (timelike) unitary and geodesic vector field E. If K and K^t denote the Gauss curvature of g and g_t respectively, then $K^t = \frac{1}{1+\varepsilon t} K$.*

Corollary 3.4. *Let (M, g_R) be a Riemannian manifold and g_t the canonical variation along a unitary vector field E with A_E normal. If Π is a plane containing E, then*

$$
\begin{aligned}
K_t(\Pi) &\leq \frac{1}{1+t} K_R(\Pi) \ \text{for } t \in (-\infty, -2) \cup (-1, 0), \\
K_t(\Pi) &\geq \frac{1}{1+t} K_R(\Pi) \ \text{for } t \in (-2, -1) \cup (0, \infty).
\end{aligned}
$$

In particular, if g_L is the standard canonical variation, then $K_L(\Pi) = -K_R(\Pi)$.

Proof. Since $A_E(X) \perp E$, we have

$$
g(A_E^2(X), X) \leq \sqrt{g(A_E(X), A_E(X))} \sqrt{g(A_E^{*\perp}(X), A_E^{*\perp}(X))} = g(A_E(X), A_E(X)).
$$

Using that E is geodesic and theorem 3.2 we get the result. $\square$

Corollary 3.5. *Let (M,g) be a Lorentzian or Riemannian manifold and g_t the canonical variation along a (timelike) unitary vector field E with A_E orthogonally normal. Then*

$$
Ric_t(E,E) = Ric(E,E) + \varepsilon t \, div \nabla_E E + \frac{t(2\varepsilon + t)}{2}(||A_E'||^2 - tr(A_E'^2)),
$$

where A_E' is the restriction to $E^\perp$ of A_E. In particular, if (M, g_R) is a Riemannian compact manifold,

$$
\begin{aligned}
\int_M Ric_t(E,E) dg_t &\leq \sqrt{|1+t|} \int_M Ric(E,E) dg_R \ \text{for } t \in (-\infty, -2) \cup (0, \infty), \\
\int_M Ric_t(E,E) dg_t &\geq \sqrt{|1+t|} \int_M Ric(E,E) dg_R \ \text{for } t \in (-2, -1)
\end{aligned}
$$

and if g_L is the standard canonical variation, then $\int_M Ric_L(E,E) dg_L = \int_M Ric(E,E) dg_R$.

Proof. From theorem 3.2 we have

$$
\begin{aligned}
Ric_t(E,E) & = Ric(E,E) + t\big(\varepsilon(div\,\nabla_E E - \varepsilon g(\nabla_E \nabla_E E, E)) - g(\nabla_E E, \nabla_E E)\big) \\
& \quad + \frac{t(2\varepsilon + t)}{2}\left(||A'_E||^2 - tr(A'^2_E)\right) \\
& = Ric(E,E) + \varepsilon t div\,\nabla_E E + \frac{t(2\varepsilon + t)}{2}\left(||A'_E||^2 - tr(A'^2_E)\right).
\end{aligned}
$$

For the second part, we take into account that the canonical volume forms of g_t and g are related by $dg_t = \sqrt{|1+t|}dg_R$ and that $||A'_E||^2 - tr(A'^2_E) \geq 0$ because A_E is orthogonally normal. $\square$

Theorem 3.6. *Let (M,g) be a Riemannian or Lorentzian manifold and g_t the canonical variation along a (timelike) unitary vector field E. If $X, Y \in E^\perp$, then*

$$
\begin{aligned}
g_t(R^t_{XY}Y, X) & = g(R_{XY}Y, X) \\
& \quad + \frac{t}{1+\varepsilon t}\Big(g(A_E(X), X)g(A_E(Y), Y) - g(A_E(X), Y)g(A_E(Y), X) \\
& \quad - \frac{4 + 3\varepsilon t}{4}d\omega(X,Y)^2\Big).
\end{aligned}
$$

Proof. From equation (2.3),

$$
\begin{aligned}
g(R^t_{XY}Y - R_{XY}Y, X) & = g((\nabla_X D^t)(Y,Y), X) - g((\nabla_Y D^t)(X,Y), X) \\
& \quad + g(D^t(X, D^t(Y,Y)), X) - g(D^t(Y, D^t(X,Y)), X).
\end{aligned}
$$

We compute each term using proposition 2.3. Ther fisrt one.

$$
\begin{aligned}
g((\nabla_X D^t)(Y,Y), X) & = -\frac{t}{2(1+\varepsilon t)}\left(L_E g\right)(Y,Y)g(E, \nabla_X X) - tg(E, \nabla_X Y)d\omega(Y,X) \\
& = \frac{t}{1+\varepsilon t}g(\nabla_Y E, Y)g(X, \nabla_X E) + tg(Y, \nabla_X E)g(\nabla_Y E, X) \\
& \quad - tg(\nabla_X E, Y)^2.
\end{aligned}
$$

The second term.

$$
\begin{aligned}
g((\nabla_Y D^t)(X,Y), X) & = -\frac{t}{2(1+\varepsilon t)}\left(L_E g\right)(X,Y)g(E, \nabla_Y X) - \frac{t}{2}g(E, \nabla_Y X)d\omega(Y,X) \\
& = \frac{t}{2}g(\nabla_Y E, X)\left(\frac{1}{1+\varepsilon t}\left(L_E g\right)(X,Y) + d\omega(Y,X)\right) \\
& = \frac{t(2+\varepsilon t)}{2(1+\varepsilon t)}g(\nabla_Y E, X)^2 - \frac{\varepsilon t^2}{2(1+\varepsilon t)}g(\nabla_X E, Y)g(\nabla_Y E, X).
\end{aligned}
$$

The third one vanishes. Using corollary 2.4, the last one is $g(D^t(Y, D^t(X,Y)), X) = \frac{\varepsilon t^2}{4(1+\varepsilon t)}\left(g(\nabla_Y E, X)^2 - g(Y, \nabla_X E)^2\right)$.

Now,

$$
\begin{aligned}
g_t(R^t_{XY}Y, X) - g(R_{XY}Y, X) & = \frac{t}{1+\varepsilon t}g(\nabla_X E, X)g(\nabla_Y E, Y) \\
& \quad + \frac{t}{2(1+\varepsilon t)}\left((2 + 3\varepsilon t)g(\nabla_X E, Y)g(\nabla_Y E, X) - \frac{4 + 3\varepsilon t}{2}(g(\nabla_X E, Y)^2 + g(\nabla_Y E, X)^2)\right) \\
& = \frac{t}{1+\varepsilon t}\left(g(\nabla_X E, X)g(\nabla_Y E, Y) - g(\nabla_X E, Y)g(\nabla_Y E, X) - \frac{4 + 3\varepsilon t}{4}d\omega(X,Y)^2\right).
\end{aligned}
$$

$\square$

Example 3.7. *A Riemannian manifold (M, g_R) is called of quasi-constant sectional curvature if there exists a unitary vector field $E \in \mathfrak{X}(M)$ such that the sectional curvature of any plane only depends on the basepoint and the angle between the plane and E, [6, 9]. On the other hand, a Lorentzian manifold (M, g_L) is called infinitesimal null isotropy if there exists a timelike unitary vector field $E \in \mathfrak{X}(M)$ such that the lightlike sectional curvature respect to E only depends on the basepoint, [15, 16]. After theorems 3.2 and 3.6 we can easily check both definitions are equivalent in their respective settings.*

Indeed. Suppose that (M, g_R) is a Riemannian manifold of quasi-constant sectional curvature. If we call ω and θ the one forms metrically equivalent to E and $\nabla_E E$ respectively, then it is known that $\nabla_X^R E = \lambda X$, $d\omega(X, Y) = 0$ and $\left(\nabla_X^R \theta\right)(Y) - \theta(X)\theta(Y) = \gamma g_R(X, Y)$, being $X, Y \in E^\perp$ and λ, γ functions on M, [9]. Take g_L the standard canonical variation along E. Using above formulas and theorems 3.2 and 3.6, it is easy to show that, in (M, g_L), the sectional curvature of planes orthogonal to E and planes containing E only depends on the basepoint, which implies that it is infinitesimal null isotropy. In the same manner, we can check that if (M, g_L) is a Lorentzian manifold infinitesimal null isotropy, then the standard canonical variation is a Riemannian manifold of quasi-constant sectional curvature.

Corollary 3.8. *Let (M, g) be a Riemannian or Lorentzian manifold and g_t the canonical variation along a (timelike) unitary vector field E with A_E orthogonally normal. Given $X \in E^\perp$,*

$$
\begin{aligned}
Ric_t(X, X) &= Ric(X, X) - \frac{t}{1 + \varepsilon t} g(R_{XE}E, X) + \frac{t}{1 + \varepsilon t} g(A_E(X), X) div\, E \\
&+ \frac{\varepsilon t^2}{1 + \varepsilon t} g(A_E^2(X), X) - t g(A_E(X), A_E(X)) \\
&+ \frac{t}{1 + \varepsilon t} \left(g(\nabla_X A_E(E), X) - \varepsilon g(A_E(E), X)^2 \right).
\end{aligned}
$$

Proof. Take $\{e_1, \dots, E_t\}$ an orthonormal basis for g_t with $E_t = \frac{1}{\sqrt{|t + \varepsilon|}} E$. Using theorem 3.2,

$$
\begin{aligned}
g_t(R_{E_t X}^t X, E_t) &= \frac{1}{|\varepsilon + t|} g(R_{XE}E, X) + \frac{t}{|\varepsilon + t|} \left(\varepsilon g(\nabla_X \nabla_E E, X) - g(\nabla_E E, X)^2 \right) \\
&+ \frac{t(2\varepsilon + t)}{2|\varepsilon + t|} \left(g(A_E(X), A_E(X)) - g(A_E^2(X), X) \right).
\end{aligned}
$$

On the other hand, from theorem 3.6,

$$
\begin{aligned}
\sum_{i=1}^{n-1} g_t(R_{e_i X}^t X, e_i) &= \sum_{i=1}^{n-1} g(R_{e_i X} X, e_i) + \\
&\frac{t}{1 + \varepsilon t} \left(g(A_E(X), X) div\, E - g(X, A_E^2(X)) - \frac{4 + 3\varepsilon t}{4} \sum_{i=1}^{n-1} d\omega(X, e_i)^2 \right).
\end{aligned}
$$

But now observe that

$$
\begin{aligned}
\sum_{i=1}^{n-1} d\omega(X, e_i)^2 &= g(A_E(X), A_E(X)) + g(A_E^{*\perp}(X), A_E^{*\perp}(X)) - 2g(X, A_E^2(X)) \\
&= 2g(A_E(X), A_E(X)) - 2g(X, A_E^2(X)),
\end{aligned}
$$

where the last equality holds because A_E is orthogonally normal. Putting all together we obtain the result. □

Corollary 3.9. *Let (M, g) be a Riemannian or Lorentzian manifold and g_t the canonical variation along a (timelike) unitary vector field E with A_E orthogonally normal. The scalar curvatures S_t and S are related by*

$$
\begin{aligned}
S_t &= S - \frac{2t}{1+\varepsilon t} Ric(E, E) + \frac{2t}{1+\varepsilon t} div \nabla_E E - tg(\nabla_E E, \nabla_E E) \\
&\quad + \frac{t}{1+\varepsilon t} \left(tr(A_E')^2 - tr(A_E'^2) \right) + \frac{\varepsilon t^2}{2(1+\varepsilon t)} \left(tr(A_E'^2) - ||A_E'||^2 \right),
\end{aligned}
$$

where A_E' is the restriction to $E^\perp$ of A_E.

Proof. We only have to apply corollaries 3.5 and 3.8. □

Example 3.10. *We can easily construct a Riemannian odd sphere with constant negative scalar curvature. Indeed, if we take g_t the canonical variation along the Hopf vector field, since it is unitary and Killing, from above corollary, $S_t = 2n(2n + 1 - t)$, which is negative for t large enough.*

Finally, we compute $g_t(R_{EX}^t X, Y)$, being $X, Y \in E^\perp$, which will be useful later.

Proposition 3.11. *Let (M, g) be a Riemannian or Lorentzian manifold and g_t the canonical variation along a (timelike) unitary vector field E. If $X, Y \in E^\perp$, then*

$$
\begin{aligned}
g_t(R_{EX}^t X, Y) &= g(R_{EX} X, Y) + \frac{t}{2} \Big(-\varepsilon(\nabla_X d\omega)(X, Y) + g(A_E(X), X)g(A_E(E), Y) \\
&\quad -2g(X, A_E(Y))g(A_E(E), X) + g(A_E(X), Y)g(A_E(E), X) \Big).
\end{aligned}
$$

Proof. From formula (6),

$$
\begin{aligned}
g_t(R_{EX}^t X - R_{EX} X, Y) &= g((\nabla_E D^t)(X, X), Y) - g((\nabla_X D^t)(E, X), Y) \\
&\quad + g(D^t(E, D^t(X, X)), Y) - g(D^t(X, D^t(E, X)), Y).
\end{aligned}
$$

As always, we use proposition 2.3 to compute each term. The first one gives us $g((\nabla_E D^t)(X, X), Y) = \frac{t}{1+\varepsilon t} g(A_E(X), X)g(A_E(E), Y) + tg(A_E(E), X)d\omega(X, Y)$. The second one.

$$
\begin{aligned}
g((\nabla_X D^t)(E, X), Y) &= Xg(D^t(E, X), Y) - g(D^t(E, X), \nabla_X Y) - \frac{\varepsilon t}{2} d\omega(\nabla_X X, Y) \\
&\quad - \frac{t}{2} g(E, \nabla_X X)d\omega(E, Y) \\
&= \frac{\varepsilon t}{2} X(d\omega(X, Y)) - \frac{\varepsilon t}{2} d\omega(X, \nabla_X Y) + \frac{t}{2} g(\nabla_X E, Y)g(\nabla_E E, X) \\
&\quad - \frac{\varepsilon t}{2} d\omega(\nabla_X X, Y) + \frac{t}{2} g(\nabla_X E, X)g(\nabla_E E, Y) \\
&= \frac{\varepsilon t}{2} (\nabla_X d\omega)(X, Y) + \frac{t}{2} \Big(g(\nabla_X E, Y)g(\nabla_E E, X) \\
&\quad + g(\nabla_X E, X)g(\nabla_E E, Y) \Big).
\end{aligned}
$$

The third is $g(D^t(E, D^t(X,X)), Y) = \frac{\varepsilon t^2}{1+\varepsilon t} g(\nabla_X E, X) g(\nabla_E E, Y)$ and the last one vanishes. $\qquad\square$

4. Standard canonical variation along a Killing vector field

Suppose that E is a Killing unitary vector field in a Riemannian manifold (M, g_R) and consider $g_L = g_R - 2\omega \otimes \omega$ the standard canonical variation along it. In this case, from formula (4), we have

$$\nabla_U^L V = \nabla_U^R V - 2\Big(\omega(U)\nabla_V^R E + \omega(V)\nabla_U^R E\Big).$$

Moreover, from corollary 2.4, E is also Killing for g_L.

The symmetric respect to E of a vector $v = \alpha E + Y$, being $Y \perp E$, is $v^* = \alpha E - Y$. The symmetric respect to E of a plane $\Pi = span(X, v)$, being $X \perp E$, is the plane given by $\Pi^* = span(X, v^*)$.

We denote $\mathcal{K}_L^E(\Pi)$ the lightlike sectional curvature of a lightlike plane Π of (M, g_L) respect to E.

Proposition 4.1. *Let (M, g_R) be a Riemannian manifold, $E \in \mathfrak{X}(M)$ a Killing unitary vector field and g_L the standard canonical variation along E.*

   (1) *If Π is a nondegenerate plane for g_L, then $K_R(\Pi^*) \leq -\cos(2\theta)K_L(\Pi)$, being θ the angle between Π and E. Moreover, the equality holds if and only if $E \in \Pi$.*
   (2) *If Π is a degenerate plane for g_L, then $2K_R(\Pi^*) \leq \mathcal{K}_L^E(\Pi)$.*
   (3) *Given $v \in TM$ it holds $Ric_R(v,v) \leq Ric_L(v^*,v^*)$ and the equality holds if and only if v is proportional to E.*
   (4) *The scalar curvatures S_R and S_L hold $S_R \leq S_L$ and the equality holds if and only if E is parallel.*

Proof. (1) Suppose $\Pi = span(X, V)$ where X and V are g_R-unitary, $V = \alpha E + Y$ and $X \perp Y \perp E$. Since E is Killing,
(7)
$$\nabla_X^R(d\omega)(X, Y) = 2\left(g_R(\nabla_X^R \nabla_X^R E, Y) - g_R(\nabla_{\nabla_X^R X}^R E, Y)\right) = -2g_R(R_{EX}^R X, Y).$$

Now, recall that g_L corresponds to the values $t = -2$ and $\varepsilon = 1$ in formula (3) and therefore, applying theorems 3.2, 3.6 and 3.11, we get

$$(8) \qquad g_L(R_{VX}^L X, V) = g_R(R_{V^*X}^R X, V^*) + 6g_R(\nabla_X^R E, Y)^2,$$

and we obtain the result.

(2) It follows from equation (8).

(3) Suppose $v = \alpha E + X$ with $X \in E^\perp$. Using corollary 3.5, $Ric_L(E, E) = Ric_R(E, E)$. From proposition 3.11 and formula (7), $Ric_L(E, X) = -Ric_R(E, X)$ and by corollary 3.8, $Ric_L(X, X) = Ric_R(X, X) - 2g_R(R_{XE}^R E, X) + 6g_R(\nabla_X^R E, \nabla_X^R E)$. Since E is Killing, $g_R(\nabla_X^R E, \nabla_X^R E) = g_R(R_{XE}^R E, X)$ and therefore

$$Ric_L(v, v) = Ric_R(v^*, v^*) + 4g_R(\nabla_X^R E, \nabla_X^R E)$$

and the result follows.

(4) Since E is Killing, $||A_E||^2 = Ric_R(E, E)$ and thus, corollary 3.9 gives us $S_R + 2Ric_R(E, E) = S_L$. The statement holds now trivially. $\qquad\square$

The existence of a Killing vector field on a Lorentzian manifold with an isolated zero implies that M has even dimension, [2]. We can proof the following related result.

Theorem 4.2. *Let (M, g_L) be a compact Lorentzian manifold with negative sectional curvature on timelike planes. If there exists a timelike unitary Killing vector field, then M has odd dimension.*

Proof. Take $g_R = g_L + 2\omega \otimes \omega$. Applying above proposition, if Π is a plane containing E, then $K_R(\Pi) = -K_L(\Pi) > 0$. Since E is also unitary and Killing for g_R (see corollary 2.4), using Berger's theorem ([4, 10]), M has odd dimension. $\square$

Example 4.3. *Since the Lorentzian Berger sphere $(\mathbb{S}^{2n+1}, g_L)$ is obtained as the standard canonical variation of the Euclidean sphere along the Hopf vector field E, we have $2 \leq \mathcal{K}_E(\Pi)$ for all degenerate plane and the scalar curvature is $S_L = 2n(2n+3)$.*

We give now an application to using Bochner techniques. Given a Killing vector field U in a compact semi-Riemannian manifold (M, g), it holds

$$\int_M ||A_U||^2 dg = \int_M Ric(U, U) dg.$$

Therefore, in the Riemannian case, we have

$$(9) \qquad 0 \leq \int_M Ric_R(U, U) dg_R.$$

Moreover, if U is nonzero everywhere and we take $\{e_1, \ldots, e_{n-1}, \frac{U}{|U|}\}$ an orthonormal basis, then

$$||A_U||^2 = \sum_{i=1}^{n-1} g_R(\nabla_{e_i}^R U, \nabla_{e_i}^R U) + \frac{1}{g_R(U, U)} g_R(\nabla_U^R U, \nabla_U^R U)$$

and so we can refine inequality (9) obtaining

$$(10) \qquad 0 \leq \int_M \left(Ric_R(U, U) - \frac{1}{g_R(U, U)} g_R(\nabla_U^R U, \nabla_U^R U) \right) dg_R.$$

In the Lorentzian case, even if U is timelike, $||A_U||^2$ and $\sum_{i=1}^{n-1} g_L(\nabla_{e_i}^L U, \nabla_{e_i}^L U)$ do not have sign and we can not obtain the inequalities (9) nor (10). Nevertheless, in [18] it is observed that $||A_E||^2 \geq 0$, where E is the unitary of U, and thus if U is timelike and Killing it holds

$$\int_M \frac{1}{g_L(U, U)} Ric_L(U, U) dg_L \leq 0.$$

Using the canonical variation we can prove a similar inequality to (10) in the Lorentzian case. First, we need some preliminaries lemmas.

Lemma 4.4. *Let (M, g) be a semi-Riemannian manifold and $E \in \mathfrak{X}(M)$ a vector field with $g(E, E) = c \in \mathbb{R} - \{0\}$. Suppose that E is orthogonally conformal, i.e, $(L_E g)(X, Y) = 2\rho g(X, Y)$ for all $X, Y \in E^\perp$, and take $\lambda \in C^\infty(M)$. Then $U = \lambda E$ is conformal if and only if*

$$\begin{aligned} E(\lambda) &= \lambda \rho, \\ cX(\lambda) &= -\lambda g(\nabla_E E, X) \text{ for all } X \in E^\perp. \end{aligned}$$

Proof. It is enough to use the formula $L_U = \lambda L_E + d\lambda \otimes \omega + \omega \otimes d\lambda$. $\square$

Lemma 4.5. *Let* (M, g_L) *be a Lorentzian manifold and* U *a timelike conformal/Killing vector field with unitary* E. *Then* U *is also conformal/Killing for the canonical variation along* E.

Proof. Just apply corollary 2.4 and above lemma. $\square$

Theorem 4.6. *Let* (M, g_L) *be a compact Lorentzian manifold and* U *a timelike Killing vector field. Then*

$$0 \leq \int_M \left(Ric_L(U,U) - \frac{2}{g_L(U,U)} g_L(\nabla_U^L U, \nabla_U^L U) \right) dg_L.$$

Proof. Call E the unitary of U ($U = \lambda E$) and consider g_t the canonical variation along E. For $1 < t < 2$, we have

$$0 \leq \int_M Ric_t(U,U) dg_t,$$

since g_t is Riemannian and U is Killing. Using corollary 3.5,

$$Ric_t(U,U) = \lambda^2 Ric_t(E,E) = \lambda^2 \left(Ric_L(E,E) - t div \nabla_E^L E + t(t-2)||A_E'||^2 \right)$$

$$= Ric_L(U,U) + \frac{2t}{\lambda^2} g_L(\nabla_U^L U, \nabla_U^L U) - t div \nabla_U^L U + t(t-2)\lambda^2 ||A_E'||^2$$

But $||A_E'||^2 \geq 0$ and $dg_t = \sqrt{|1-t|} dg_L$. Therefore

$$0 \leq \int_M \left(Ric_L(U,U) - \frac{2t}{g_L(U,U)} g_L(\nabla_U^L U, \nabla_U^L U) \right) dg_L,$$

for $1 < t < 2$. Taking $t \to 1$ we get the result. $\square$

Example 4.7. *In general, the integral (10) is not positive in the Lorentzian case. In fact, let* (N, g_0) *be a compact Riemannian manifold,* $f \in C^\infty(N)$ *a positive function and consider* $(M, g_L) = \left(N \times \mathbb{S}^1, g_0 - f^2 dt^2 \right)$. *The vector field* $U = \partial_t$ *is timelike and Killing and it is easy to show that*

$$\int_M \left(Ric_L(U,U) - \frac{1}{g_L(U,U)} g_L(\nabla_U^L U, \nabla_U^L U) \right) dg_L = - \int_M g_L(\nabla f, \nabla f) dg_L \leq 0.$$

5. Standard canonical variation along a closed vector field

Suppose that E is a closed unitary vector field in a Riemannian manifold (M, g_R) and consider $g_L = g_R - 2\omega \otimes \omega$ the standard canonical variation along it. From formula (4), we have

$$\nabla_U^L V = \nabla_U^R V + 2g_R(\nabla_U^R E, V)E.$$

In particular, the shape operators of the orthogonal leaves of E in (M, g_R) and (M, g_L) coincide.

Proposition 5.1. *Let* (M, g_R) *be a Riemannian manifold,* $E \in \mathfrak{X}(M)$ *a closed unitary vector field and* g_L *the standard canonical variation along* E.

(1) *If Π is a nondegenerate plane for g_L, then*

$$-\cos(2\theta)K_L(\Pi) = K_R(\Pi) + 2\sin^2(\theta)\left(\hat{K}_R(\mathfrak{p}(\Pi)) - K_R(\mathfrak{p}(\Pi))\right),$$

where θ is the angle between E and Π, $\hat{K}_R$ is the induced curvature on the orthogonal leaves of E and $\mathfrak{p}$ is the orthogonal projection onto $E^\perp$.
In particular, $K_L(\Pi) = -K_R(\Pi)$ for any plane containing E.

(2) *If Π is a degenerate plane for g_L, then*

$$\mathcal{K}_L^E(\Pi) = 2K_R(\Pi) + 2\left(\hat{K}_R(\mathfrak{p}(\Pi)) - K_R(\mathfrak{p}(\Pi))\right).$$

(3) *For any $v \in TM$, it holds*

$$\begin{aligned} Ric_L(v,v) &= Ric_R(v,v) + 2g_R(A_E(v),v)div_R\,E \\ &\quad - 2g_R(A_E(v),A_E(v)) - 2g_R(R_{vE}^R E, v). \end{aligned}$$

(4) $S_L = S_R + 4E(div_R\,E) + 2\left(||A_E||^2 + (div_R\,E)^2\right).$

Proof. Suppose $\Pi = span(X,V)$ where X and V are g_R-unitary, $V = \alpha E + Y$ and $X \perp Y \perp E$. Applying theorem 3.2, 3.6 and proposition 3.11, we get $g_L(R_{VX}^L X, V) = g_R(R_{VX}^R X, V) + 2\left(g_R(A_E(X), X)g_R(A_E(Y), Y) - g_R(A_E(X), Y)^2\right)$ and we can easily deduce points 1 and 2.
(3) It follows from corollary 3.5, 3.8 and proposition 3.11.
(4)Use that $Ric_R(E) + E(div_R\,E) + ||A_E||^2 = 0$ and corollary 3.9. $\qquad\square$

Theorem 5.2. *Let (M, g_R) be a Riemannian manifold, $E \in \mathfrak{X}(M)$ a closed unitary vector field and g_L the standard canonical variation along E. If E is complete and nonparallel, then there exists a point $p \in M$ such that $S_R(p) < S_L(p)$.*

Proof. Suppose that $S_L \leq S_R$ for all points in M. Using above proposition,

$$2E(div_R\,E) + (div_R\,E)^2 \leq 2E(div_R\,E) + ||A_E||^2 + (div_R\,E)^2 \leq 0.$$

Since E is complete, $div_R\,E = 0$ and $S_L = S_R + 2||A_E||^2 \geq S_R \geq S_L$. Therefore $||A_E||^2 = 0$ and E is parallel. $\qquad\square$

Now, we are interested in knowing whether the completeness is preserved by the canonical variation. In general, this does not hold, as the following examples show.

Example 5.3. *Take (L, g_0) a complete Riemann manifold and the warped product $(\mathbb{R} \times L, -dt^2 + f^2(t)g_0)$. The standard canonical variation along ∂_t is $g_R = dt^2 + f^2(t)g_0$, which is always complete. However, we can choose f such that $-dt^2 + f^2(t)g_0$ is not complete, [20].*

Example 5.4. *Consider the Lorentzian plane $(\mathbb{R}^2, g_L = -dx^2 + dy^2)$. Call*

$$f(x,y) = e^{-(x+y)}, \quad a(x,y) = \frac{1 + f(x,y)^2}{2f(x,y)}, \quad b(x,y) = \frac{1 - f(x,y)^2}{2f(x,y)}.$$

Now, $E = a\partial_x + b\partial_y$ is a timelike unitary vector field and the standard canonical variation along it is $g_R = (2a^2 - 1)dx^2 + (2b^2 + 1)dy^2 - 4ab\,dxdy$. Take $\gamma : [0, \infty) \to \mathbb{R}^2$ given by $\gamma(t) = (t,t)$. It escapes any compact subset of $\mathbb{R}^2$, but $\lim_{s\to\infty} \int_0^s \sqrt{g_R(\gamma', \gamma')} = \frac{\sqrt{2}}{2}$. Therefore, $(\mathbb{R}^2, g_R)$ is incomplete.

An important case when completeness is preserved is when E is a Killing vector field. In this case, if (M, g_R) is complete, then it is also (M, g_L). Using this, it follows that a compact Lorentzian manifold furnished with a timelike Killing vector field is complete, [19].

We can prove another case when completeness is assured. For this, we give the following definition.

Definition 5.5. *Let $\overline{M}$ be a nondegenerate hypersurface of a semi-Riemannian manifold (M, g). We say that $\overline{M}$ is strongly curved if its shape operator is semi-definite.*

Proposition 5.6. *Let (M, g_R) be a Riemannian manifold and E a complete, unitary and closed vector field. If the orthogonal leaves of E are complete and strongly curved, then (M, g_R) is complete.*

Proof. We can suppose that M is simply connected. Hence, it splits as $\big(\mathbb{R} \times L, ds^2 + g_s\big)$, where ∂_s is identified with E, L is an orthogonal leaf and g_s is a Riemannian metric on L for each $s \in \mathbb{R}$, [13].

Suppose that the shape operator of the orthogonal leaves is semi-definite negative. Let $v \in T_x L$ and take $V(s) = (0_s, v_x)$. Then, since $[V, E] = 0$, we have $\frac{d}{ds} g_R(V(s), V(s)) = 2g_R(\nabla_V^R E, V) \geq 0$. Thus $g_s(v, v)$ is an non decreasing function.

Call d the distance induced by g_R in M, d_s the distance induced by g_s in L and take $\{p_n = (t_n, x_n)\}$ a Cauchy sequence. Since $d(p_n, p_m) \geq |t_n - t_m|$ we have that $\{t_n\}$ converges to, say, t_0 and we can suppose that $|t_n - t_0| \leq \delta$ for all $n \in \mathbb{N}$ and certain $\delta \in \mathbb{R}$. Given $0 < \varepsilon < \delta$ there is $n_0 \in \mathbb{N}$ such that $d(p_m, p_n) < \varepsilon$ for $m, n \geq n_0$. Let $\gamma(s) = (s(t), x(t))$ be a curve with $\gamma(0) = p_m$ and $\gamma(1) = p_n$. We can suppose that $|s(t) - t_0| \leq 2\delta$ since on the contrary case, $L(\gamma) > \delta > d(p_m, p_n)$. Now,

$$
L(\gamma) \;=\; \int_0^1 \sqrt{s'(t)^2 + g_{s(t)}(x'(t), x'(t))} \geq \int_0^1 \sqrt{g_{s^*}(x'(t), x'(t))} \geq d_{s^*}(x_n, x_m),
$$

where $s^* = t_0 - 2\delta$ and therefore $d_{s^*}(x_n, x_m) \leq d(p_n, p_m) < \varepsilon$. Hence $\{x_n\}$ is a Cauchy sequence in (L, g_{s^*}) and so it converges. $\square$

Theorem 5.7. *Let (M, g_L) a Lorentzian manifold and E a complete timelike unitary and closed vector field. If the orthogonal leaves of E are complete and strongly curved, then the standard canonical variation g_R along E is also complete.*

Proof. As it was said at the beginning of this section, the shape operators coincide in (M, g_L) and (M, g_R), thus orthogonal leaves of E are strongly curved in (M, g_R) too. Applying above proposition we obtain that g_R is complete. $\square$

6. Lightlike hypersurfaces

Let (M, g_L) be a Lorentzian manifold, $E \in \mathfrak{X}(M)$ a timelike unitary vector field and $\overline{M}$ a lightlike hypersurface. We can fix a lightlike vector field $\xi \in \mathfrak{X}(\overline{M})$ with $g_L(E, \xi) = \frac{1}{\sqrt{2}}$ and consider the screen distribution given by $\mathcal{S} = T\overline{M} \cap E^\perp$. Given $U, V \in \mathfrak{X}(\overline{M})$, the second fundamental form of $\overline{M}$ is $B(U, V) = -g_L(\nabla_U^L \xi, V)$. The hypersurface $\overline{M}$ is totally geodesic if $B = 0$ and totally umbilic if $B = \rho g$ for certain $\rho \in C^\infty(\overline{M})$.

We can decompose

$$\nabla_U^L V = \overline{\nabla}_U^L V + B(U,V)N,$$
$$\nabla_U^L \xi = -\tau(U)\xi - A^*(U),$$

where $\overline{\nabla}_U^L V \in T\overline{M}$, $A^*(U) \in \mathcal{S}$, τ is a one form and $N = \sqrt{2}E + \xi$ is the transverse vector field to $\overline{M}$. Recall that $\nabla_\xi^L \xi = -\tau(\xi)\xi$ and N is lightlike and orthogonal to $\mathcal{S}$. Moreover, $\tau(\xi) = \sqrt{2}g_L(\nabla_\xi^L E, \xi)$.

Observe that $X_0 = \frac{1}{\sqrt{2}}E + \xi$ is orthogonal to E and therefore $\xi = -\frac{1}{\sqrt{2}}E + X_0$ and $N = \frac{1}{\sqrt{2}}E + X_0$, i.e., N is the symmetric of $-\xi$ respect to E.

If we consider g_R the standard canonical variation, then $g_R(\xi,\xi) = g_R(N,N) = 1$ and N is g_R-normal to $\overline{M}$. Call $\mathbb{I}$ the second fundamental form of $\overline{M}$ in (M, g_R).

Proposition 6.1. *Given $X, Y \in \mathcal{S}$, it holds*

$$\mathbb{I}(X,Y) = \left(B(X,Y) - \frac{1}{\sqrt{2}}\left(L_E g_L\right)(X,Y) \right)N,$$
$$\mathbb{I}(X,\xi) = -g_L(\nabla_{E+\sqrt{2}\xi}^L E, X)N,$$
$$\mathbb{I}(\xi,\xi) = -\left(2g_L(\xi, \nabla_E^L E) + \tau(\xi)\right)N.$$

Proof. By proposition 2.3, $g_R(D(X,Y), N) = -\frac{1}{\sqrt{2}}\left(L_E g\right)(X,Y)$ and thus we have

$$g_R(\mathbb{I}(X,Y), N) = g_R(\nabla_X^L Y, N) - \frac{1}{\sqrt{2}}\left(L_E g\right)(X,Y) = B(X,Y) - \frac{1}{\sqrt{2}}\left(L_E g\right)(X,Y).$$

On the other hand $g_R(D(X,\xi), N) = -\sqrt{2}g_L(\nabla_\xi^L E, X) - g_L(\nabla_E^L E, X)$ and since $g_R(\nabla_X^L \xi, N) = 0$, we get $\mathbb{I}(X,\xi) = -g_L(\nabla_{E+\sqrt{2}\xi}^L E, X)N$.

Finally, $g_R(D(\xi,\xi), N) = -\sqrt{2}g(\xi, \nabla_N^L E) = -\left(2g_L(\xi, \nabla_E^L E) + \tau(\xi)\right)$, and since $g_R(\nabla_\xi^L \xi, N) = 0$, we have $\mathbb{I}(\xi,\xi) = -\left(2g_L(\xi, \nabla_E^L E) + \tau(\xi)\right)N$. $\square$

If E is parallel and $\overline{M}$ is totally geodesic, then it is also totally geodesic in (M, g_R), but however, if $\overline{M}$ is umbilic, then it is not umbilic in (M, g_R).

Recall that the lightlike mean curvature of $\overline{M}$ is given by $H_L = \sum_{i=1}^{n-2} B(e_i, e_i)$ where $\{e_1, \ldots, e_{n-2}\}$ is an orthonormal basis of $\mathcal{S}$ and the mean curvature of $\overline{M}$ as hypersurface of (M, g_R) by $H_R = \sum_{i=1}^{n-1} g_R(\mathbb{I}(v_i, v_i), N)$ being $\{v_1, \ldots, v_{n-1}\}$ a g_R orthonormal basis of $T\overline{M}$.

Proposition 6.2. *Let (M, g_L) be a Lorentzian manifold, $E \in \mathfrak{X}(M)$ a timelike unitary vector field and g_R the standard canonical variation along it. Take $\overline{M}$ a lightlike hypersurface.*

- $H_R = H_L - \sqrt{2}div_L E + \tau(\xi)$.
- *If $\overline{M}$ is compact and orientable, then $\int_{\overline{M}} H_L dg_R = 0$.*

Proof. If $\{e_1, \ldots, e_{n-2}\}$ an orthonomal basis in $\mathcal{S}$, then $\{e_1, \ldots, e_{n-2}, \xi\}$ is a g_R-orthonormal basis of $\overline{M}$. Therefore, using proposition 6.1,

$$H_R = H_L - \sqrt{2}\sum_{i=1}^{n-2} g_L(\nabla_{e_i}^L E, e_i) - 2g_L(\xi, \nabla_E^L E) - \tau(\xi).$$

But now, observe that $\{e_1, \ldots, e_{n-2}, \sqrt{2}\xi + E, E\}$ is an orthonormal basis and so

$$
\begin{aligned}
div_L E &= \sum_{i=1}^{n-2} g_L(\nabla_{e_i}^L E, e_i) + g_L(\nabla_{\sqrt{2}\xi + E}^L E, \sqrt{2}\xi + E) \\
&= \sum_{i=1}^{n-2} g_L(\nabla_{e_i}^L E, e_i) + \sqrt{2}\tau(\xi) + \sqrt{2}g_L(\nabla_E^L E, \xi).
\end{aligned}
$$

For the second point, since $g_R(D(e_i, \xi), e_i) = 0$ (corollary 2.4), we have $div_R^{\overline{M}}\xi = -H_L$. $\qquad\square$

We can easily obtain the following generalization of theorem 8 in [3].

Corollary 6.3. *Let (M, g_L) be a Lorentzian manifold, E a timelike unitary Killing vector field and $\overline{M}$ a lightlike hypersurface. Consider g_R the standard canonical variation along E. The lightlike mean curvature of $\overline{M}$ vanishes if and only if the mean curvature of $\overline{M}$ as hypersurface of (M, g_R) vanishes.*

In a time-orientable and orientable Lorentzian manifold satisfying the null convergence condition ($Ric(u, u) \geq 0$ for all lightlike vector u) any compact lightlike hypersurface with nonpositive (or nonnegative) mean curvature is totally geodesic. In fact, we only have to take into account second point of proposition 6.2 and the well-known Raychaudhuri equation (see for example [7, 12, 17]). An standard application of the Raychaudhuri equation also gives us that a lightlike hypersurface is totally geodesic if the null convergence condition and the completeness of lightlike geodesics of the hypersurface are assumed. Observe that this result is not applicable to above situation because the compactness does not ensure, in general, the completeness of the lightlike geodesics of the hypersurface.

We can obtain other results ensuring that a compact lightlike hypersurface is totally geodesic under curvature hypotheses. First, we need the following lemma.

Lemma 6.4. *Let (M, g_L) be a Lorentzian manifold and $U \in \mathfrak{X}(M)$ a timelike Killing vector field with unitary E. If $\lambda = |U|$, then for any $X \perp E$ we have*

$$
(11) \qquad g_L(R_{XE}^L E, X) = \frac{1}{\lambda} g_L(\nabla_X^L \nabla \lambda, X) + g_L(\nabla_X^L E, \nabla_X^L E).
$$

If U is a Killing vector field and $\overline{M}$ a compact hypersurface of a Riemannian manifold (M, g_R) with normal unitary N, then

$$
(12) \qquad \int_{\overline{M}} g_R(U, N) H_R dg_R = 0.
$$

Moreover, if $\overline{M}$ has constant mean curvature, then

$$
(13) \qquad \int_{\overline{M}} g_R(U, N)\|S\|^2 - Ric_R(N, \overline{U})dg_R = 0,
$$

where $S(X) = -\nabla_X^R N$ is the shape operator of $\overline{M}$ and $\overline{U}$ is the g_R-projection of U onto $\overline{M}$, [24].

Theorem 6.5. *Let (M, g_L) be a Lorentzian manifold, $U \in \mathfrak{X}(M)$ a timelike Killing vector field and $\overline{M}$ a compact lightlike hypersurface. Take E the unitary of U, $\lambda = |U|$ and $\xi \in \mathcal{X}(\overline{M})$ a lightlike vector field with $g_L(\xi, E) = \frac{1}{\sqrt{2}}$. Call X_0 the orthogonal projection of ξ onto $E^\perp$ and N the symmetric of $-\xi$ respect to E. If*

- $H_L - g_L(\nabla \ln \lambda, \xi)$ *has sign.*
- $0 \leq Ric_L(N, \xi) + \triangle \ln \lambda - 4g_L(\nabla^L_{X_0} E, \nabla^L_{X_0} E).$

Then, $\overline{M}$ is totally geodesic.

Proof. First, observe that $\nabla^L_E E = \nabla \ln \lambda$ and $\tau(\xi) = -g_L(\xi, \nabla \ln \lambda)$. Take g_R the canonical variation along $E = \frac{U}{\lambda}$. We know that U is also Killing for g_R (lemma 4.5) and it is easy to show that $g_R(U, N) = \frac{\lambda}{\sqrt{2}}$. Applying formula (12), we have $\int_{\overline{M}} \lambda H_R dg_R = 0$. But, from proposition 6.2, $H_R = H_L - g_L(\nabla \ln \lambda, \xi)$ which has sign. Therefore, $H_R = 0$ and since the projection of U onto $\overline{M}$ is given by $-\frac{\lambda}{\sqrt{2}} \xi$, formula (13) gives us

$$\int_{\overline{M}} \lambda ||S||^2 dg_R = - \int_{\overline{M}} \lambda Ric_R(N, \xi) dg_R.$$

Using corollary 3.5 and 3.8 and lemma 6.4,

$$
\begin{aligned}
Ric_R(N, \xi) &= -\frac{1}{2} Ric_R(E, E) + Ric_R(X_0, X_0) \\
&= Ric_L(N, \xi) + \triangle \ln \lambda - 6g_L(\nabla^L_{X_0} E, \nabla^L_{X_0} E) \\
&\quad + 2\left(g_L(R^L_{X_0 E} E, X_0) - \frac{1}{\lambda} g_L(\nabla^L_{X_0} \nabla \lambda, X_0)\right) \\
&= Ric_L(N, \xi) + \triangle \ln \lambda - 4g_L(\nabla^L_{X_0} E, \nabla^L_{X_0} E).
\end{aligned}
$$

Thus, $||S||^2 = 0$ and from proposition 6.1, $\overline{M}$ is a totally geodesic lightlike hypersurface of (M, g_L). $\qquad\square$

Corollary 6.6. *Let (M, g_L) be a Lorentzian manifold, $E \in \mathfrak{X}(M)$ a timelike unitary Killing vector field and $\overline{M}$ a compact lightlike hypersurface. Take $\xi \in \mathcal{X}(\overline{M})$ a lightlike vector field with $g_L(\xi, E) = \frac{1}{\sqrt{2}}$ and N the symmetric of $-\xi$ respect to E. If*

- H_L *has sign.*
- $0 \leq Ric_L(N, \xi) + 2K_L(span(\xi, N)).$

Then $\overline{M}$ is totally geodesic.

Proof. From lemma 6.4, $g_L(\nabla^L_{X_0} E, \nabla^L_{X_0} E) = g_L(R^L_{X_0 E} E, X_0) = -\frac{1}{2} K^L(span(\xi, N)).$ $\qquad\square$

References

[1] C. Antidogbé, Scalar curvature on lightlike hypersurfaces, Appl. Sci. 11 (2009) 9-18.

[2] J.K. Beem, P.E. Ehrlich and S. Markvorsen, Timelike isometries and Killing vector fields, Geom. Dedicata 26 (1988) 247-258.

[3] A. Bejancu, A. Ferrández and P. Lucas, A new viewpoint on geometry of a lightlike hypersurface in a semi-Euclidean space, Saitama Math. J. 16 (1998) 31-38.

[4] M. Berger, Trois remarques sur les variétés Riemanniennes a courbure positive, C. R. Acad. Sci. Paris 263 (1966) 76-78.

[5] A. Besse, Einstein manifolds, Springer-Verlag, Berlin, 1987.

[6] V. Boju and M. Popescu, Espaces courbure quasi-constante, J. Diff. Geom. 13 (1978) 373-383.

[7] K.L. Duggal and A. Giménez, Lightlike hypersurfaces of Lorentzian manifolds with distinguished screen distribution, J. Geom. Phys. 55 (2005) 107-122.

[8] J.L. Flores, M.A. Javaloyes and P. Piccione, Periodic geodesics and geometry of compact Lorentzian manifolds with a Killing vector field, Math. Z. 267 (2011) 221-233.

[9] G. Ganchev and V. Mihova, Riemannian manifolds of quasi-constant sectional curvatures, J. reine angew. Math. 522 (2000), 119-141.

[10] V.N. Berestovski and Y.G. Nikonorov, Killing vector fields of constant length on Riemannian manifolds, Sib. Math. J. 49 (2008) 395-407.
[11] S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time, Cambridge University Press, London-New York, 1973.
[12] G.J. Galloway, Maximum principles for null hypersurfaces and null splitting theorems, Ann. Henri Poincaré 1 (2000) 543-567.
[13] M. Gutierrez and B. Olea, Global decomposition of a Lorenzian manifold as a Generalized Robertson-Walker space, Diff. Geom. Appl. 27 (2009) 146-156.
[14] M. Gutiérrez, F.J. Palomo and A. Romero, A Berger-Green type inequality for compact Lorentzian manifolds, Trans. Amer. Math. Soc. 354 (2002) 4505-4523.
[15] S. G. Harris, A triangle comparison theorem for Lorentz manifolds, Indiana Univ. Math. J. 31 (1982) 289-308.
[16] H. Karcher, Infinitesimale Charakterisierung von Friedmann-Univeren, Arch. Math. 38 (1982) 5864.
[17] D.N. Kupeli, On null submanifolds in spacetimes, Geom. Dedicata 23 (1) (1987) 3351.
[18] A. Romero and M. Sánchez, Bochner's techniques in Lorentzian manifold, Pacific J. Math. 186 (1998) 141-148.
[19] A. Romero and M. Sánchez, Completeness of compact Lorentz manifolds admiting a timelike conformal Killing field, Proc. Amer. Math. Soc. 123 (1995) 2831-2833.
[20] M. Sánchez, On the geometry of Generalized Robertson-Walker Spacetimes: Geodesics, Gen. Relativity Gravitation 30 (1998) 915-932.
[21] V.V. Reddy, R. Sharma and S. Sivaramakrishnan, Lorentzian metric induced from a background Riemannian metric, Int. J. Pure Appl. Math. 47 (2008) 343-351.
[22] V.V. Reddy, R. Sharma and S. Sivaramakrishnan, Spacetimes through HawkingEllis construction with a background Riemannian metric, Class. Quantum Grav. 24 (2007) 3339-3345.
[23] M. Strake, Curvature increasing metric variation, Math. Ann. 276 (1987) 633-641.
[24] K. Yano, Closed hypersurfaces with constant mean curvature in a Riemannian manifold, J. Math. Soc. Japan 17 (1995) 333-340.
[25] U. Yurtsever, Test fields on compact spacetimes, J. Math. Phys. 31 (1990) 3064-3078.

Departamento de Matemáticas, Instituto E.S. Torre Almenara, Mijas. Málaga. Spain.

E-mail address: benji@agt.cie.uma.es